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A SIMPLE MODEL TO PREDICT THE FUNDAMENTAL FREQUENCY OF THE REINFORCED CONCRETE DOME STRUCTURE OF A DOUBLE-SHELL UNDERGROUND TANK

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ABSTRACT

A simple, classic, theoretical model based on shallow-shell theory is presented for predicting the fundamental frequency of the reinforced concrete dome structure of Hanford high-level waste Tank 241-SY-101. Theoretical results are compared with two separate finite-element analysis (FEA) models: (1) an axisymmetric model showing only the reinforced concrete dome structure that incorporates a fixed boundary condition at the dome haunch region and equipment and soil mass above the dome; and (2) an axisymmetric model of the complete tank structure, including equipment, soil mass, and fluid mass to the 400-in. waste level. An estimate of the fundamental frequency is developed further for a degraded material model based on an FEA of the tank structure subjected to a transient pressurization from a hydrogen burn event. The degraded material state model provides insight into the natural frequency reduction after material damage has occurred, such as concrete cracking and steel yielding.

1.0. INTRODUCTION

In performing steady-state, transient-dynamic, or impact analyses of structures, an important property of the system to be considered is the natural frequency of vibration. Dynamic amplification, or resonance, may occur if the frequency of a harmonic or unsteady applied load is at, or near, the natural frequency of the structure. A pure impulse load also may amplify the natural response of a structure based on the duration of the impulse load.

Accident analyses[1-3] for high-level waste (HLW) Tank 241-SY-101 (Tank 101-SY) (see Fig. 1) that might cause structural amplification include hydrogen burns, dropped loads on the tank dome (such as during pump removal, causing potential severe impacts), and seismic motion with fluid sloshing. These types of events may cause severe damage to the structure if the right conditions exist for dynamic amplification.

In this analysis, we will consider only the fundamental mode of vibration. The fundamental frequency of a structure is the frequency at the lowest natural mode of vibration. Although a structure may exhibit many modes of vibration because of the numerous possible degrees of freedom, the lowest natural mode, or lowest natural frequency, is termed the fundamental frequency.

During hydrogen burns, which are of greatest concern to the dome structure, the period of the transient pressure may coincide with the fundamental harmonics of the dome. As such, it is imperative to determine the natural vibration of the tank structure to prevent dynamic amplifications and possible failure. The effect on the tank structure of falling objects also must be considered to create potential dynamic amplifications. Of concern here is a drop accident of the hydrogen mixer pump during removal or installation.

Previous analyses of dropped loads have neglected considering the natural frequency of the structure or the harmonics of the impact loads. Pump impact analyses performed by Westinghouse Hanford Company (WHC) and ADVENT Engineering have shown that the period of impact for the dropped mixer pump is ~100 to 200 ms. This would correspond to a forcing frequency between 5 and 10 Hz. Therefore, it is imperative that the dome's fundamental frequency be determined, thus maintaining a basis for conducting continuous external tank operations and minimizing potential impact failure to the dome.

The following analysis presents a simple method for determining the fundamental frequency of the dome structure of Tank 101-SY and compares the results to finite-element solutions.

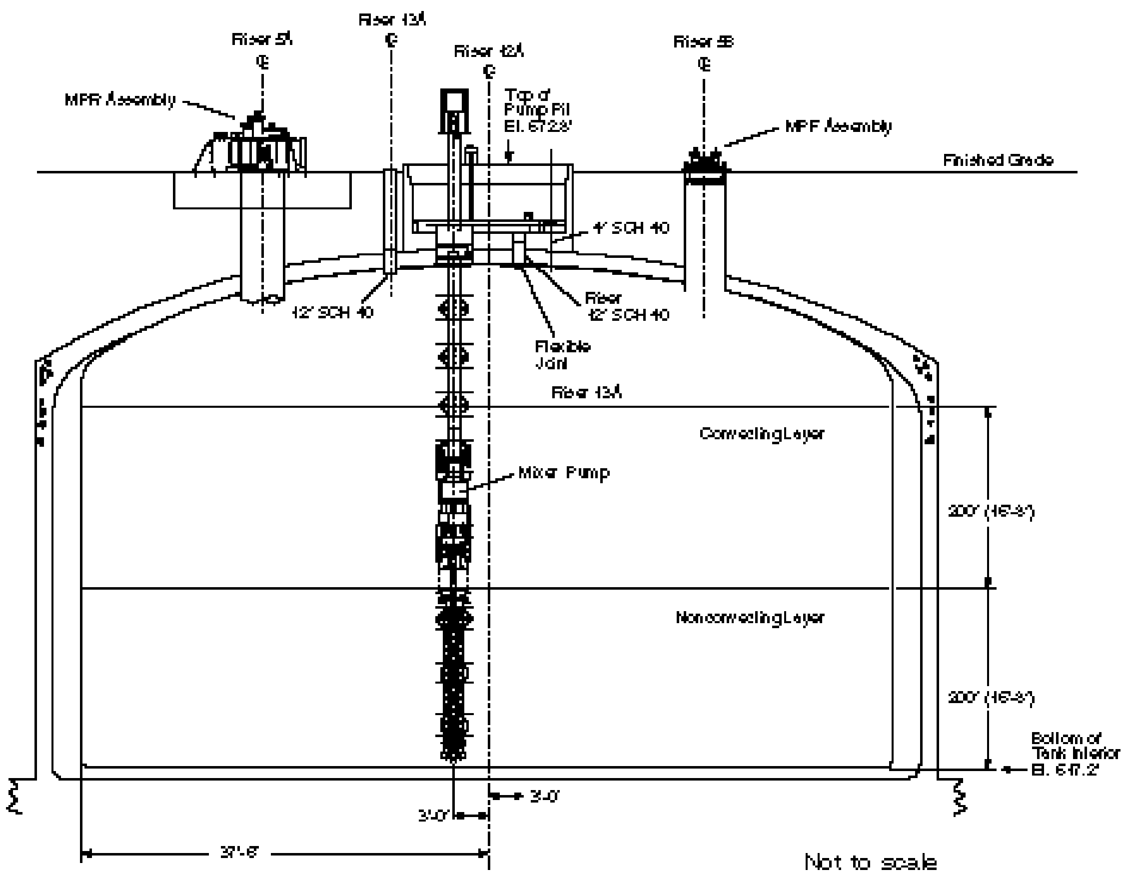

Fig. 1. Cross-sectional view of Tank 241-SY-101.

2.0. ANALYSIS METHOD AND ASSUMPTIONS

Two methods will be used to determine and compare the natural frequency of the dome structure: (1) classic theoretical vibration analysis[4-7] based on the shallow-shell theory; and (2) finite-element analysis (FEA) of an axisymmetric representation of the dome, as well as a complete representation of the tank structure. Figure 2 provides a cross-sectional view of the tank structure with dome dimensions used in the analysis.

2.1. Classic Vibration Analysis Theory

The classic vibration analysis theory predicts natural frequencies of structures or systems based on the classic solutions of the differential equations of motion.[4-8] In this report, simple single-degree-of-freedom (SDOF) systems will be used to determine the fundamental frequency of the Tank 101-SY reinforced concrete dome structure based on the shallow-shell theory.

In many engineering problems, the response of a structure to an applied loading, whether static or dynamic, will be in the elastic range of the material. Designing of equipment, structures, pressure vessels, etc., generally is performed with the criteria of maintaining pure elasticity of the material. That is, the design of the structure is based on maintaining stresses and strains that are developed from expected operational loading at or below the linear elastic limit of the material. Elastic analysis methodology thus simplifies the design but, more importantly, the analysis.

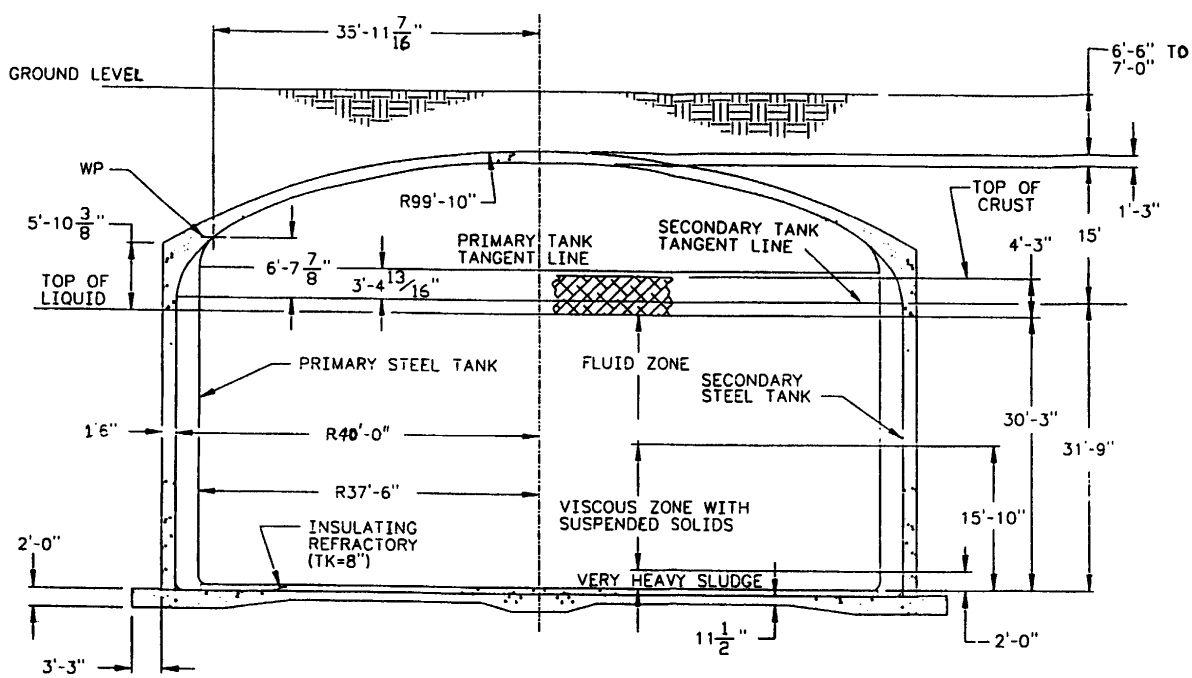

Fig. 2. Tank 101-SY cross-sectional dimensions.

In vibration analysis, the same rigor (or lack thereof) is applied to structural systems that are known to remain within the linear elastic limit of the material. Using this philosophy, vibration analysis of structures also may be simplified when the behavior or response remains within the linear elastic limit. The natural vibration modes of a structure can be extracted by solving the differential equations of motion. The simple SDOF system of Fig. 3 responds according to Newton's Law of motion, where the sum of the forces equals the mass times the acceleration of the mass.

From Newton's Law,

$$\sum F = m\ddot{x} \quad . \tag{1}$$

The motion of an undamped SDOF system from Fig. 3 under free vibrations is governed by a general homogeneous second-order linear Ordinary Differential Equation[9-10] (ODE) as

$$m\ddot{x} + kx = 0 \quad , \tag{2}$$

where the displacement amplitude of the structure is defined by

$$x(t) = A\cos\omega_n t + B\sin\omega_n t \tag{3}$$

and

$$\omega_n^2 = k / m \text{ or } \omega_n = \sqrt{\frac{k}{m}} \quad , \tag{4}$$

where

ω_n = natural frequency, (rad / s) and
k = spring stiffness (lb / in.).

2.2. Finite-Element Model

We used the finite-element model (FEM) for the Tank 101-SY safety assessment,[1] using the ABAQUS/Standard[11] finite-element numerical code. Much of the description relative to the model geometry, specific modeling assumptions, and boundary conditions is contained in subsequent Los Alamos National Laboratory (LANL) reports.[2,3] Therefore, no additional modeling information will be presented here. Figure 4 shows the axisymmetric FEM of Tank 101-SY.

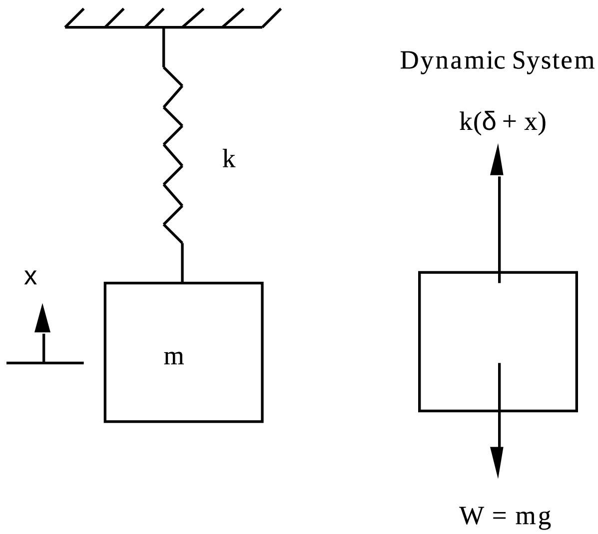

Fig. 3. Simple SDOF oscillator.

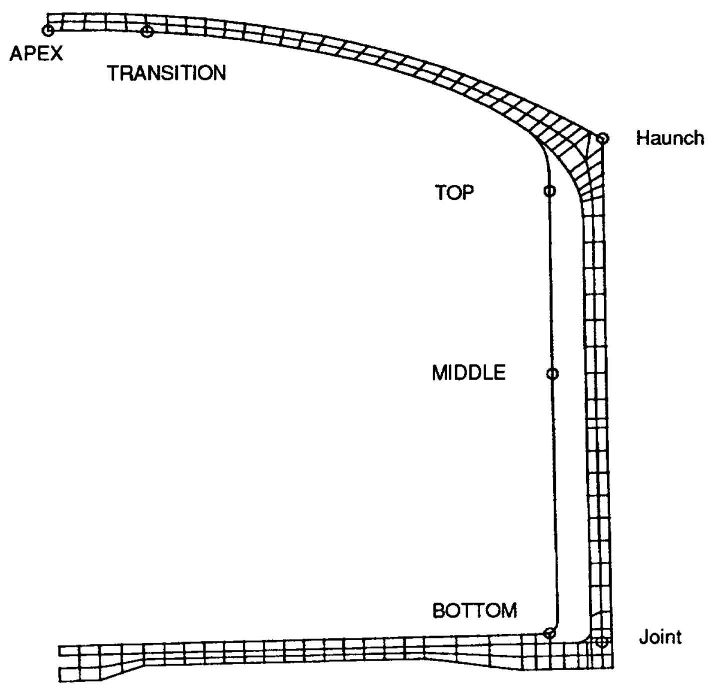

Fig. 4. Axisymmetric FEM of Tank 101-SY.

2.3. Material Properties

Material properties used in the FEM analysis and classic theoretical solutions are shown in Tables 1 and 2 and Fig. 5. Additional information not contained in this report may be found in LANL safety assessment documents[1] and supplemental reports.[2,3]

TABLE 1
REINFORCED CONCRETE PARAMETERS

Modulus E_s **(psi)**	**Poisson's Ratio** ν	**Yield Strain** ε_y	**Plastic Strain** ε_p	**Max Strain** ε'_{max}
4.696E+6	0.15	6.388E-4	1.917E-3	2.555E-3

TABLE 2
MINIMUM SPECIFIED AND BEST-ESTIMATE[a] STEEL PROPERTIES[b]

Material	**Specification**	**Yield Strength (ksi)**	**Ultimate Strength (ksi)**
Steel Liner	ASTM A-516 Grade 65	35.0 (Min)[a] 44.0 (B-E)[b]	65.0 (Min) 71.0 (B-E)
Steel Rebar (Shear Ties)	ASTM A-615 Grade 40	40.0 (Min) 49.0 (B-E)	70.0 (Min) 76.0 (B-E)
Steel Rebar (Main Rebars)	ASTM A-615 Grade 60	60.0 (Min) 71.0 (B-E)	90.0 (Min) 110.0 (B-E)

[a]Min = minimum.
[b]B-E = best estimate.

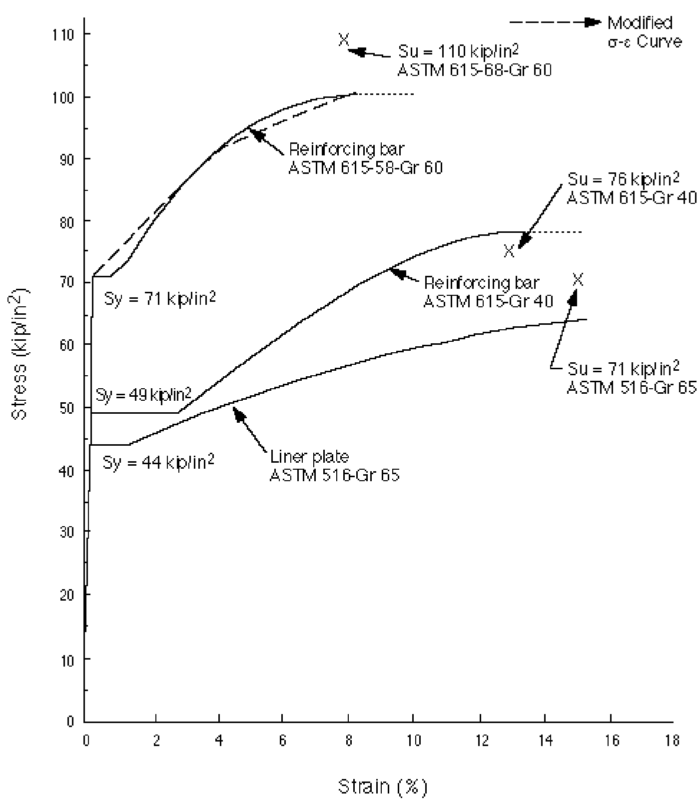

Fig. 5. Engineering stress-strain relationship of American Society for Testing and Materials (ASTM) steels.

Specific reinforcing-bar details relative to the main portion of the dome and the haunch region are shown in Figs. 6 and 7, respectively. These details are used in estimating the required composite material strength for the classic solution method.

3.0. RESULTS

3.1. Classic Vibration Analysis Based on the Shallow-Shell Theory

The natural frequency of a structure is governed by the mass and spring stiffness of the system according to the classic solution of an SDOF system as

$$\omega_n = \sqrt{\frac{k}{m}} \ , \tag{5}$$

where

k = the spring stiffness (lb / in.),
m = the mass (lb-s^2 / in.), and
ω_n = the natural frequency (rad / s).

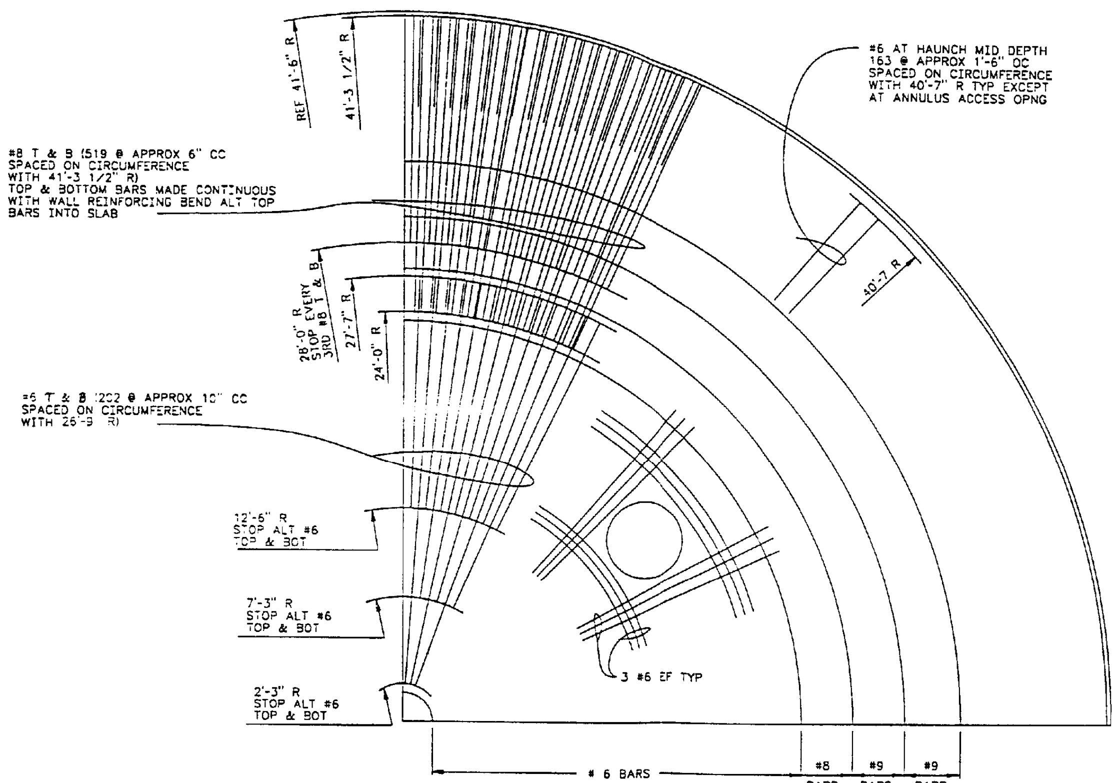

Fig. 6. Dome reinforcing-bar details.

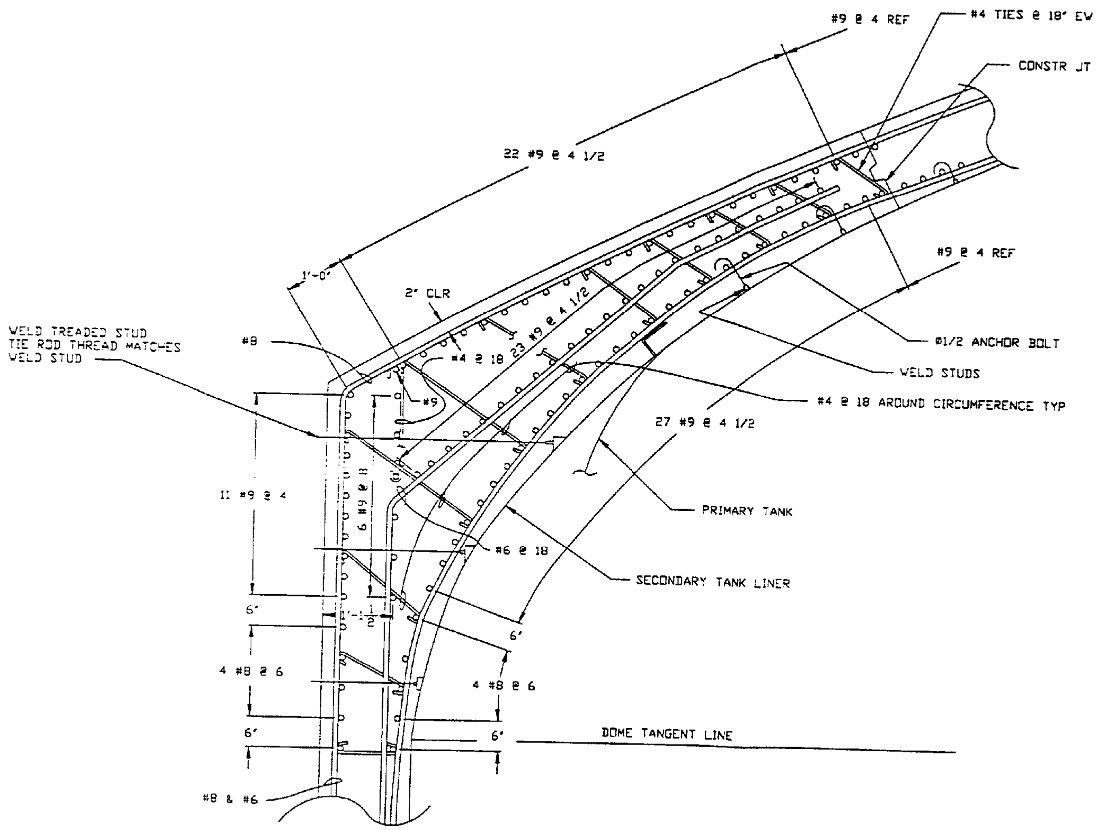

Fig. 7. Dome-haunch-region reinforcing-bar details.

Classic vibration analysis for spherically curved panels, based on the shallow-shell theory, has shown[6] that the natural frequency is a function of an equivalent flat-plate frequency as

$$\omega_{ij_{\substack{Shallow\\Spherical\\Shell}}} = \sqrt{\omega^2_{ij_{\substack{Flat\\Plate}}} + \frac{E}{\mu R_S^2}} \;, \tag{6}$$

where

E = the modulus of elasticity (lb / in.^{2}),
R_S = the radius of the curvature of the shell (in.),
μ = the mass density of material (mass / in.^{3}),
$\omega_{ij_{\substack{Flat\\Plate}}}$ = the natural frequency of equivalent flat plate (rad / s), and
ij = the mode indices.

The natural frequency of the flat plate refers to an "equivalent" flat plate that is formed from the projection of the spherical shell on a plane surface, which has the same geometric parameters as the spherical shell. That is, the equivalent flat plate must have the same thickness, homogeneous isotropic material, analogous homogeneous boundary conditions, and projected lateral dimensions as the spherical shell. The shallow-shell dome geometry of Tank 101-SY projects a shape of an equivalent circular flat plate. However, the dome geometry is much closer to a semielliptical shallow shell than to a spherical shallow shell. The radius of curvature for a semielliptical geometry is derived from the equation of an ellipse as

$$R_s = \sqrt{\frac{R_c^4}{h^2} + \left(1 - \frac{R_c^2}{h^2}\right)x^2} \ , \tag{7}$$

where

R_c = the radius of the cylindrical shell connecting to the dome (in.),
R_s = the radius of the curvature of the semielliptical shell (in.),
h = the height of the shell from the knuckle tangent to the apex (in.), and
x = the horizontal distance from the centerline to a point on the surface (in.).

For the radius of curvature at the apex of the dome where $x = 0$, Eq. (6) reduces to

$$R_s = \frac{R_c^2}{h} \ , \tag{8}$$

where

R_c = 450 in. and
h = 180 in.

Therefore,

R_s = 1125 in.

Reinforced concrete has a weight density (γ) of ~150 lb/ft^3. Therefore, the mass density is

$$\mu_{rc} = \frac{\gamma}{g} \ ,$$

where

g = the gravitational constant = 386.4 in./s^2 and
μ_{rc} = the reinforced concrete mass density (lb-s^2/in.^{4}).

Therefore,

$$\mu_{rc} = 2.2465\text{E-4 lb-s}^2/\text{in.}^4.$$

Although shallow shells react to applied loading in a membrane and bending response, the predominant response for a “breathing-type” mode of vibration is the membrane response. As such, the effective modulus of elasticity will be computed based on a composite volume of concrete and reinforcing bars in a membrane-type behavior. This procedure would not be appropriate if the predominant mode of vibration is bending. Otherwise, the plate or shell bending stiffness of the composite section would need to be determined in lieu of the membrane stiffness. The modulus of elasticity for normal structural concrete is[12]

$$E_c = 33\left(w^{1.5}\right)\sqrt{f'_c} \ , \tag{9}$$

where

w = the weight of the concrete (lb / ft^3),
w = 150 lb / ft^3,
f'_c = the compressive strength of the concrete after 28 days (lb / in.^{2}), and
f'_c = 4500 psi.

Therefore,

$$E_c = 60625\sqrt{f'_c} \ , \tag{10}$$

or

$$E_c = 4.066\text{E+6 psi} \ .$$

The compressive yield strength is taken as 50% of the maximum compressive strength (f'_c), or 3000 psi. The yield and maximum strains at the maximum compressive strength are shown in Table 1 and are derived as

$$\varepsilon_y = \frac{0.5\, f'_c}{E_s} \tag{11}$$

and

$$\varepsilon'_{max} = \frac{2 f'_c}{E_s} \ ,$$

or

$$\varepsilon'_{max} = \frac{2\sqrt{f'_c}}{33w^{1.5}} \ . \tag{12}$$

The amount of plastic strain at the maximum strain (ε'_{max}) is

$$\varepsilon_p = \varepsilon'_{max} - \varepsilon_y \ . \tag{13}$$

Figure 8 shows a differential volume of the reinforced concrete dome, which has reinforcing bars. Through-thickness rebar in the meridional direction comprises approximately eight No.-9 bars, whereas the radial direction comprises approximately six No.-9 bars. This amounts to 14 No.-9 bars in the differential volume of Fig. 8. The modified or effective elastic modulus of the composite section is merely a function of the individual moduli of elasticity of concrete and steel and their respective volumes.

(Note: No.-9 bars have a diameter of 1.128 in. and a cross-sectional area of 1.0 in^2.)

$$E_{cs} = \frac{V_c E_c + V_s E_s}{V_t} \ , \tag{14}$$

where

E_{cs} = the effective modulus of elasticity (psi),
E_c = the concrete modulus of elasticity (psi),

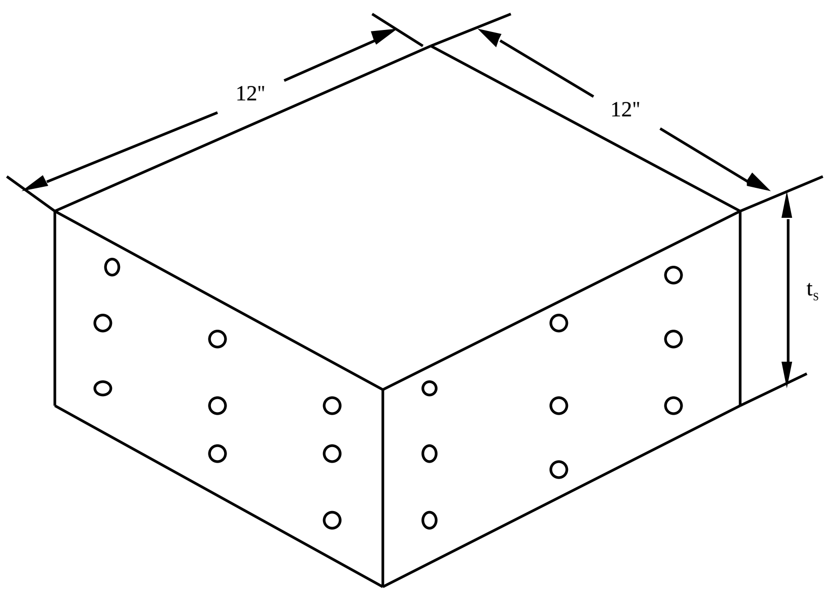

Fig. 8. The volume of a dome, which has reinforcing bars.

E_s = the steel modulus of elasticity (psi),
V_c = the concrete volume (in.^{3}),
V_s = the steel volume (in.^{3}), and
V_t = the total volume of section (in.^{3}).

The total volume of the section in Fig. 8 that is 12 in. wide by 12 in. long by 15 in. thick is

$$V_t = 2160 \text{ in.}^2 .$$

The volume of steel in this section is

$$V_s = NA_bL ,$$

where

N =the number of steel bars,
A_b =the cross-sectional area of steel bar, and
L =the length of the bars.

Therefore,

$$V_s = 168 \text{ in.}^2 ,$$

and the volume of concrete is

$$V_c = 1992 \text{ in.}^2 .$$

Then, the effective modulus of elasticity is

$$E_{cs} = 5.807\text{E+6 psi} .$$

Calculating the fundamental frequency of an equivalent flat, circular plate based on the modified modulus of elasticity is

$$\omega_{ij} = \frac{\lambda_{ij}^2}{R_c^2}\sqrt{\frac{E_{cs}t_s^3}{12\eta\left(1-\nu^2\right)}} , \tag{15}$$

where

t_s =the thickness of the plate, or shallow shell (in.);
t_s =15 in.;
ν =Poisson’s ratio ≈ 0.15;
η =the mass per unit area of the plate;

$\eta = \mu_{rc} t_s$; and
λ_{ij} =the dimensionless parameter, which is a function of the mode indices.

The following values of λ_{ij}^2 are derived for flat, circular plates with simply supported and fixed boundary conditions for the fundamental modes of vibration as

$$\lambda^2_{ij_{\substack{Simple\\Support}}} = 4.977$$

and

$$\lambda^2_{ij_{\substack{Clamped\\Support}}} = 10.22 .$$

Then, the fundamental frequency of the flat, circular plate is

$$\omega_{ij_{\substack{Simple\\Support}}} = 17.31 \text{ rad/s}$$

and

$$\omega_{ij_{\substack{Clamped\\Support}}} = 35.54 \text{ rad/s}.$$

With the flat-plate solutions, the overall fundamental frequency of the dome structure then is

$$\omega_{ij_{\substack{Shallow\\Spherical\\Shell}}} = \sqrt{\omega^2_{ij_{\substack{Flat\\Plate}}} + \frac{E_{cs}}{\mu R_S^2}} \ . \tag{16}$$

The fundamental frequencies in cycles per second, i.e., hertz (Hz), for the dome structure without the soil mass are shown in Table 3. The two separate boundary conditions for simple support and clamped support offer very little difference. This indicates that the membrane action of the shallow shell is predominant in the frequency calculations and that the flat-plate correction is negligible.

Clearly, these frequencies seem quite high because they represent the natural frequencies of a dome structure and its own mass without the added mass above the dome. However, the dome supports ~7.72M lb of soil, reinforced concrete pads, concrete pump pit, cover block, mixer pump, and other equipment. As such, we must account for the overall mass above the dome in determining the actual fundamental frequency of the structure (see Table 4). Conservatively, we modify the mass density term of the shallow-shell equation to include the overall mass above the dome, yet maintain all other parameters the same.

TABLE 3
NATURAL FREQUENCIES FOR DOME MASS

Spherical Shell Boundary Condition	Frequency	
	ω_{ij} **(rad/s)**	f_{ij} **(Hz)**
Simple Support	143.9	22.9
Clamped Support	147.3	23.4

TABLE 4
NATURAL FREQUENCIES FOR DOME AND ADDED MASS

Spherical Shell Boundary Condition	Frequency	
	ω_{ij} **(rad/s)**	f_{ij} **(Hz)**
Simple Support	47.48	7.56
Clamped Support	48.57	7.73

This method of "smearing" the mass density of the soil, concrete pads, concrete pump pit, cover block, mixer pump, and additional equipment assumes that the overall mass is attached to the dome and is concentrated within the dome structure.

Clearly, this is an oversimplification of the problem; nevertheless, this solution will provide a very good estimate of the dome frequency:

$\mu_{s/e}$ = the combined mass density of the soil and equipment.

Because the soil extends over the 80-ft diameter of the dome, the mass density assumed to be concentrated within the thickness of the dome shell is approximated as

$$\mu_{s/e} = \frac{W_{s/e}}{\frac{\pi}{4} D_c^2 t_s g} \;, \tag{17}$$

where

$W_{s/e}$ = the weight of soil and equipment above dome (lb),
$W_{s/e}$ = 7.72E6 lb,
D_c = the diameter of the tank (in.),
D_c = 960 in.,
t_s = the thickness of the dome shell (in.), and
g = the gravitational constant = 386.4 in / s^2.

Therefore,

$$\mu_{s/e} = 1.8402\text{E-3 lb-s}^2/\text{in.}^4$$

The combined mass densities of reinforced concrete dome and the soil/equipment are

$$\mu_c = \mu_{rc} + \mu_{s/e}\ , \tag{18}$$

and thus,

$$\mu_c = 2.065\text{E-3 lb-s}^2/\text{in.}^4$$

Then the overall fundamental frequency, including the equivalent flat-plate portion, is estimated as

$$\omega_{ij_{\substack{Shallow\\Spherical\\Shell}}} = \sqrt{\omega^2_{ij_{\substack{Flat\\Plate}}} + \frac{E_{cs}}{\mu_c R_s^2}}\ ,$$

or

$$\omega_{ij_{\substack{Shallow\\Spherical\\Shell}}} = \left[\frac{\lambda^4_{ij}}{R_c^4}\left(\frac{E_{cs}t_s^3}{12\eta\left(1-\nu^2\right)}\right) + \frac{E_{cs}}{\mu_c R_s^2}\right]^{\frac{1}{2}}\ . \tag{19}$$

Table 5 lists the fundamental frequencies of the dome structure with and without the soil mass and equipment above the dome. Again, as stated previously, it is apparent that the support boundary conditions (simple vs clamped) offer very little difference in the fundamental frequency calculation. The implication is that the membrane effects of the shallow shell outweigh the flat-plate correction factor.

3.2. Axisymmetric FEA of the Tank

The results of the Tank 101-SY axisymmetric analysis for the model shown in Fig. 4 are given in Table 6.

The results appear to agree within 1.5% with the classic shallow-shell theoretical results of Table 5. However, this tank model accounts for the complete structure of the dome and tank, which includes additional overall flexibility. Thus, it would seem logical that the fundamental frequency calculated by FEM would be somewhat lower than the theoretical result (as noted in Table 7).

3.3. Axisymmetric FEA of the Dome Structure

An FEM of the dome structure only, without the complete tank, was used to solve the eigenvalue problem. This process determined how the theoretical results would compare to a model only of the dome. Because the classic method specifically

TABLE 5
SUMMARY RESULTS

	Fundamental Dome Frequency Boundary Conditions	
Case	**Simple Support**	**Clamped Support**
Dome Mass without Soil Mass	22.9 Hz	23.4 Hz
Dome Mass and Smeared Soil Mass	7.56 Hz	7.73 Hz

TABLE 6
EIGENVALUE OUTPUT

Mode Number	Eigenvalue	Frequency (rad/s)	Generalized Mass
1	2231.0	47.23	15.8.
2	2563.3	50.63	3882.5
3	2714.2	52.10	3689.0

TABLE 7
SUMMARY RESULTS

Loading Case	w_{ij} (rad/s)	f_{ij} (Hz)
Dome Mass without Soil Mass	124.1	19.75
Dome Mass and Smeared Mass of Soil	52.0	8.28

assumes the flexibility of the dome structure, the FEM of the dome should be a direct comparison.

The fundamental frequency appears to be slightly higher in the classic solution of the dome model because the spring stiffness of the tank is not represented. As such, the additional flexibility afforded by the tank is not considered, thereby overpredicting the natural frequency of the system. These results agree with the theoretical results within 8%.

3.4. FEM Analysis of the Degraded Material Model

An FEM analysis of the full tank model was performed, imposing a severe burn pressure transient (see Fig. 9) that produced excessive material degradation. At the instant in time that the analysis developed peak dome apex velocities, the analysis was aborted and an eigenvalue extraction step was included. Figure 10 shows the velocity-time history of the dome apex. At 0.15 s into the transient, the analysis was ended and the eigenvalue was extracted.

Because the analysis determined extensive material degradation in the form of concrete cracking, primary liner yielding, rebars yielding, etc., the revised structural stiffness matrix is used for the eigenvalue extraction step. As such, the natural frequencies resulting from the analysis are based on a degraded material state.

Table 8 itemizes the first three modes of vibration, providing an estimate of the dome structure's fundamental frequency (Mode 1). The generalized mass is an estimate of the mass quantity that participates during the first mode of vibration. The resulting fundamental frequency under a degraded material state is approximately 0.54 Hz, with approximately 100% of the total tank and overburden mass participating.

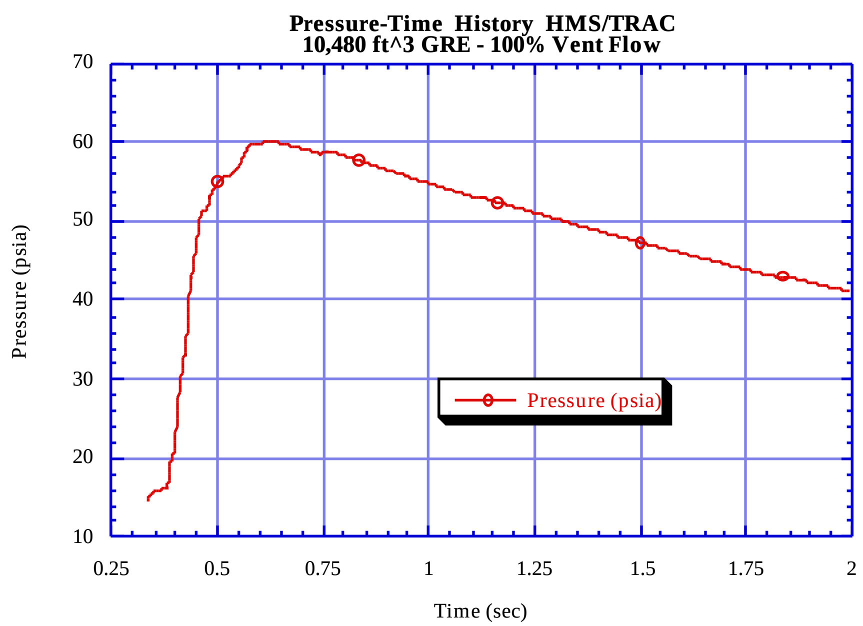

Fig. 9. Pressure-time history for Tank 101-SY burn.

TABLE 8
EIGENVALUE OUTPUT

Mode Number	Eigenvalue	Frequency (rad/s)	Generalized Mass
1	11.495	3.3905	34,184.0
2	900.26	30.004	7042.0
3	1309.5	36.186	1180.0

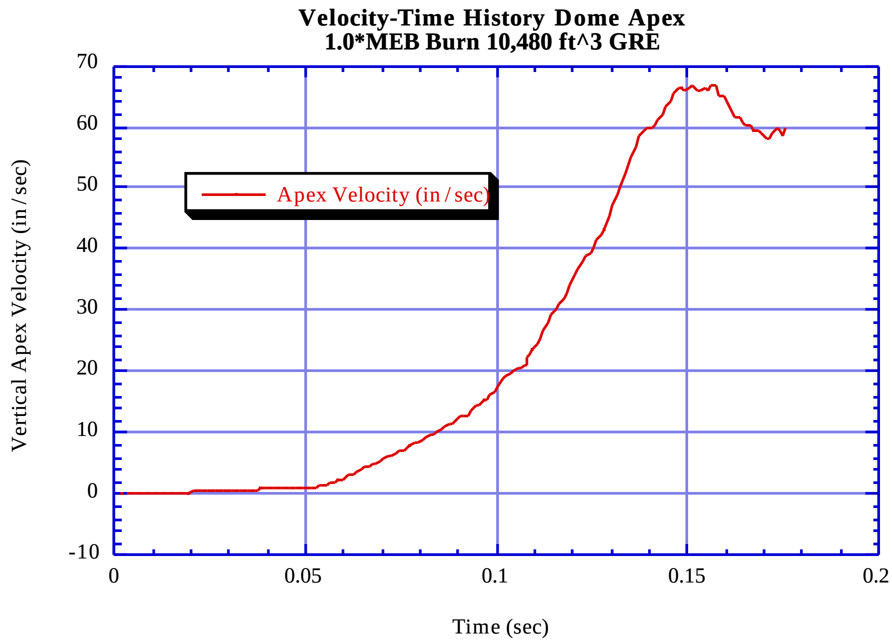

Fig. 10. Velocity-time history of dome apex.

4.0. SUMMARY AND CONCLUSIONS

This report presents a simple method to determine the fundamental frequency of a reinforced-concrete, spherical, shallow-shell structure based on classic vibration analysis theory and compares the results to the FEM. Results of a degraded material model, based on FEM analysis, also are presented to show that the fundamental frequency of the tank structure degrades because of failure mechanisms, such as plasticity, concrete cracking, and steel rebar debonding from the concrete.

The results show a very accurate correlation between the classic theoretical and FEM analyses.

We conclude that the classic theoretical model provides an accurate and simple solution to a complex problem and that this method can be used as a first approximation to assess the fundamental frequency of an underground reinforced concrete dome structure.

5.0. ACKNOWLEDGMENTS

This report was prepared before the untimely death of my friend and colleague Rich Davidson. Thanks, Rich, for all your help and constant pursuit of excellence.

6.0. REFERENCES

1. L. H. Sullivan, "A Safety Assessment for Proposed Pump Mixing Operations to Mitigate Episodic Gas Releases in Tank 241-SY-101: Hanford Site, Richland, Washington," Los Alamos National Laboratory report LA-UR-92-3196, Rev. 14, (May 1995).

2. E. A. Rodriguez, T. A. Butler, and W. D. Birchler, "Structural Response of an Underground Double-Shell Tank to a Hydrogen Burn Event," Los Alamos National Laboratory report LA-UR-95-941 (March 1995).

3. E. A. Rodriguez, R. F. Davidson, and M. L. Walsh, "Comparison of ABAQUS and EPIC Nonlinear Transient Dynamic Models for a Hydrogen Burn Event of Hanford Site HLW Tank 241-SY-101," Los Alamos National Laboratory report LA-UR-95-3441 (October 1995).

4. S. P. Timoshenko, D. H. Young, and W. Weaver, *Vibration Problems in Engineering* (John Wiley & Sons, Inc., New York, New York, 4th Edition, 1974).

5. C. M. Harris, *Shock & Vibration Handbook* (McGraw-Hill Book Co., New York, New York, 3rd Edition, 1988).

6. R. D. Blevins, *Formulas for Natural Frequency and Mode Shape* (Krieger Publishing Co., Malabar, Florida, 1993).

7. M. Paz, *Structural Dynamics—Theory and Computation* (Chapman & Hall Publishing, 3rd Edition, New York, New York, 1991).

8. J. M. Biggs, *Introduction to Structural Dynamics* (McGraw-Hill Book Co., New York, New York, 1964).

9. W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems* (John Wiley & Sons, Inc., New York, New York, 2nd Edition, 1969).

10. C. R. Wylie and L. C. Barret, *Advanced Engineering Mathematics* (McGraw-Hill Book Co., New York, New York, 5th Edition, 1982).

11. D. Hibbitt, B. Karlsson, and P. Sorensen, *ABAQUS User's Manual* (HKS Inc., Providence, Rhode Island, 1994).

12. P. M. Ferguson, *Reinforced Concrete Fundamentals* (John Wiley & Sons, Inc., New York, New York, 3rd Edition, 1973).